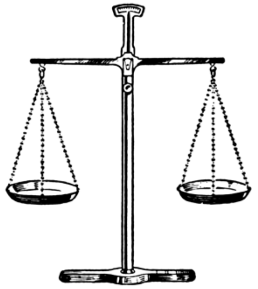
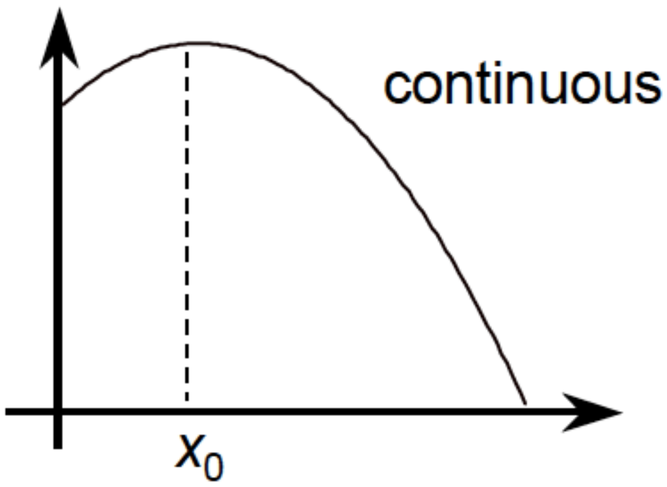
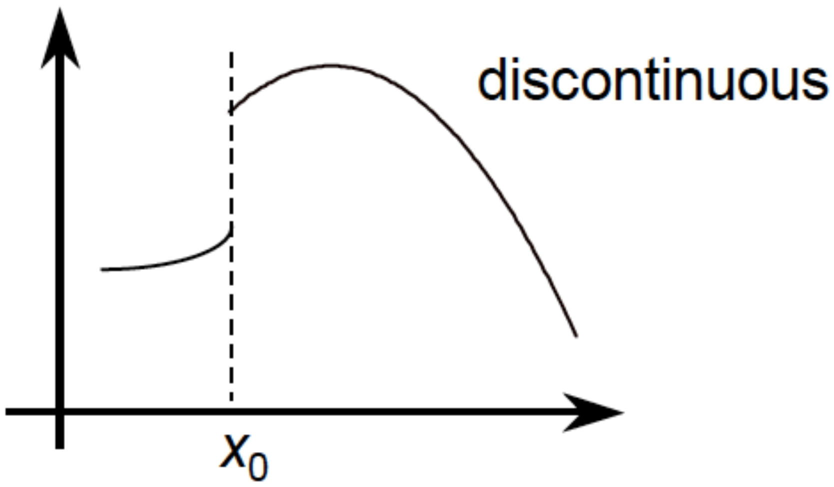


Unconstrained Non Linear Programming

Continuity





Functions can be continuos but their derivative not be

Continuity

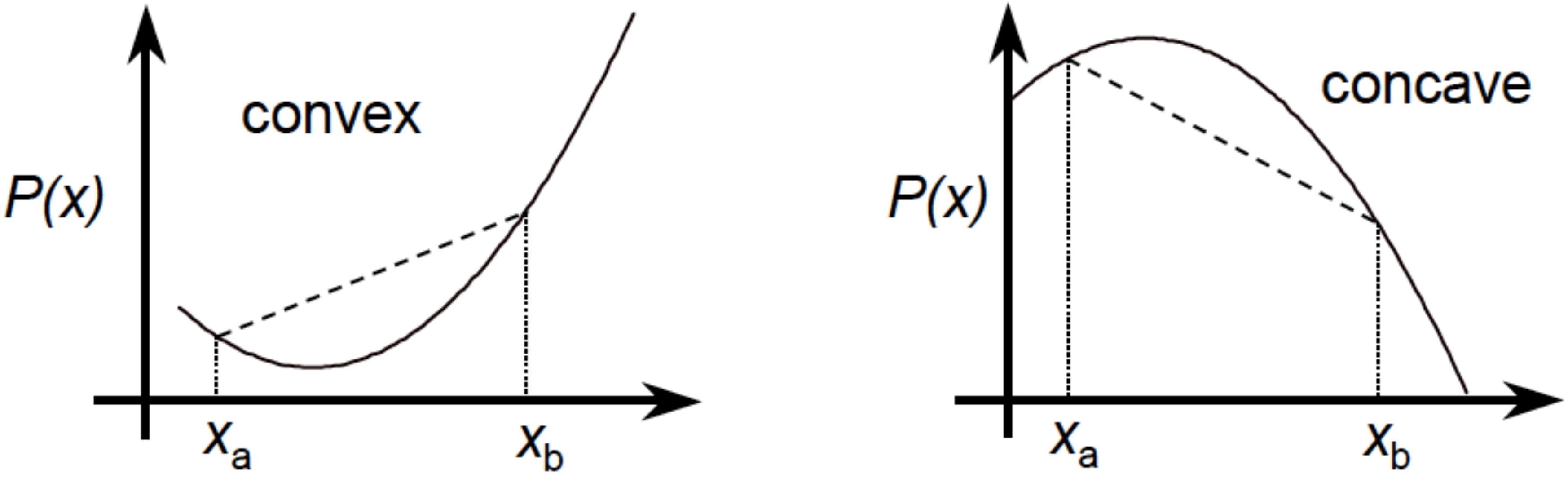
A function $P(x)$ is continuous at a point x_0 iff:

$$P(x_0)\ exists$$

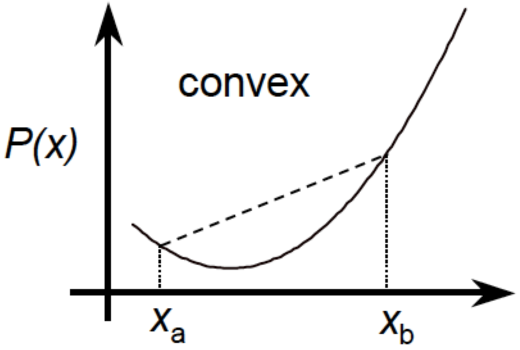
and

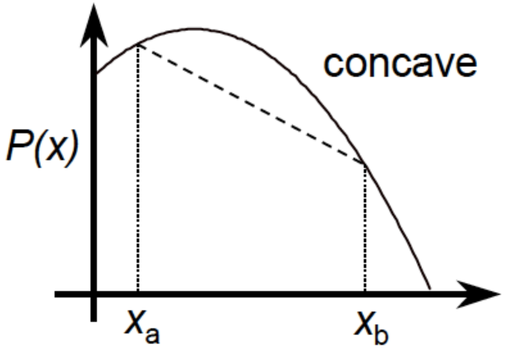
$$\lim_{x \to +x_0} P(x_0) \equiv \lim_{x \to -x_0} P(x_0) \equiv P(x_0)$$

Convexity



Convexity





This can be expressed mathematically as:

i) for a convex function and $\forall\ \theta \in [0,1]$

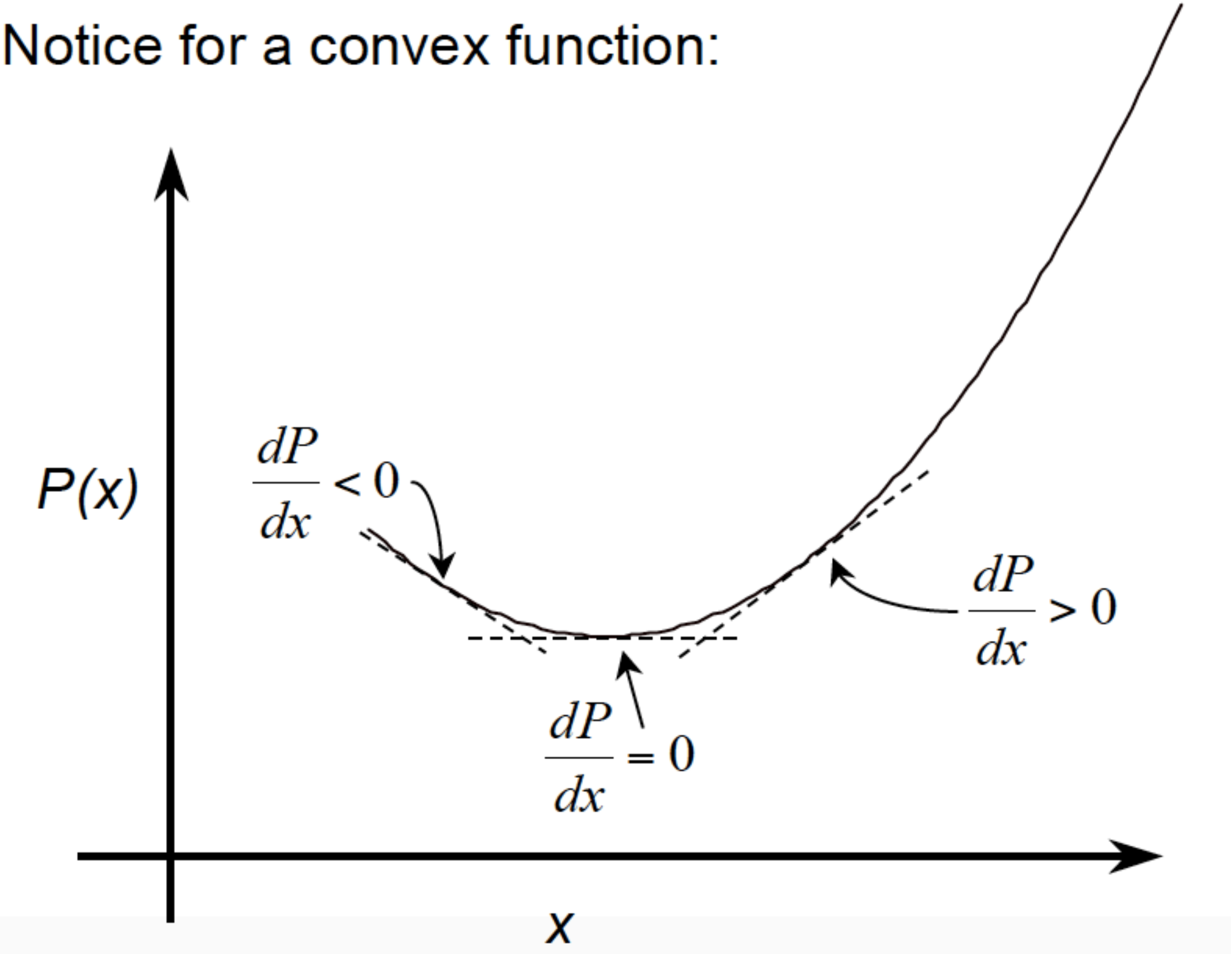
$$P(\theta\, x_a + [1-\theta] x_b) \le \theta\, P(x_a) + [1-\theta] P(x_b)$$

ii) for a concave function and $\forall\ \theta \in [0,1]$

$$P(\theta\, x_a + [1-\theta] x_b) \ge \theta\, P(x_a) + [1-\theta] P(x_b)$$

Convexity

Notice for a convex function:



Convexity

A point x* is a (local) minimum iff

$$P(x^*) \leq P(x) \quad \forall\, x \in [x_a, x_b]$$

If P(x) is convex then x* is also a global minimum

Necessary and sufficient contidions for an optimum

Recall that for a twice continuously differentiable function $P(x)$, the point x^* is an optimum iff:

$$\left.\frac{dP}{dx}\right|_{x^*} = 0$$

stationarity

and:

$$\left.\frac{d^2P}{dx^2}\right|_{x^*} > 0$$

a minimum

$$\left.\frac{d^2P}{dx^2}\right|_{x^*} < 0$$

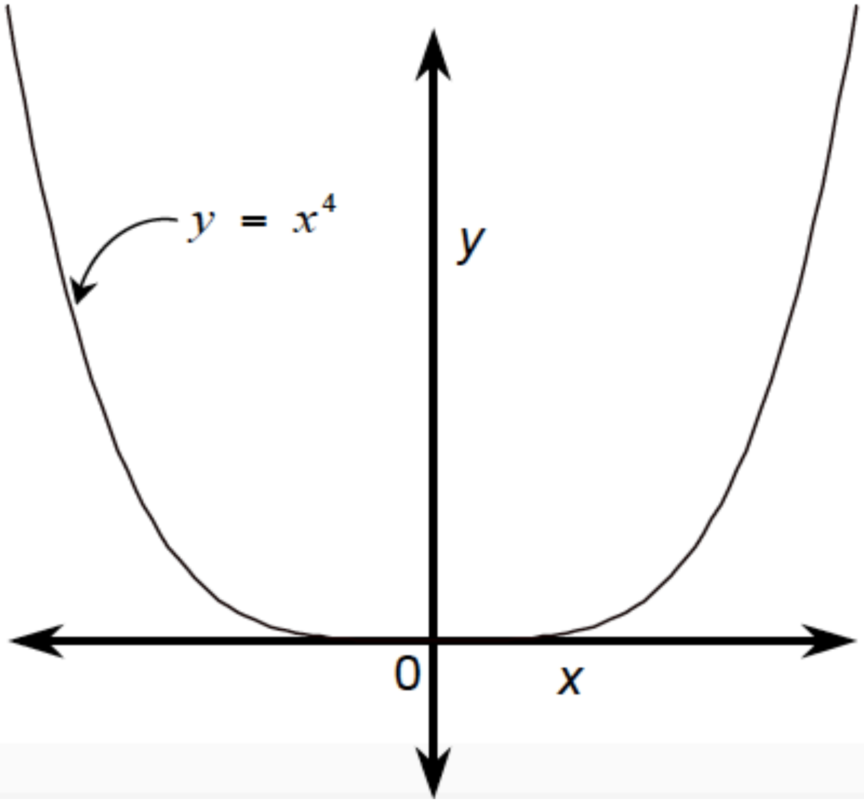
a maximum

Necessary and sufficient contidions for an optimum

What happens when the second derivative is zero?
Consider the function:

$$y = x^4$$

Which we know to have an minimum at x = 0.



Necessary and sufficient contidions for an optimum

At the point x=0:

$$\left.\frac{dy}{dx}\right|_{x=0} = 4x^3 = 0$$

stationarity

$$\left.\frac{d^2y}{dx^2}\right|_{x=0} = 12x^2 = 0$$

a minimum?

$$\left.\frac{d^3y}{dx^3}\right|_{x=0} = 24x = 0$$

$$\left.\frac{d^4y}{dx^4}\right|_{x=0} = 24 > 0$$

Necessary and sufficient contidions for an optimum

We need to add something to our optimality conditions. If at the stationary point x^* the second derivative is zero we must check the higher-order derivatives. Thus, for the first non-zero derivative is odd, that is:

$$\left.\frac{d^n P}{dx^n}\right|_{x^*} \neq 0 \quad \text{where} \quad n \in \{3,5,7,\ldots\}$$

Then x^* is an inflection point. Otherwise, if the first higher-order, non-zero derivative is even:

$$\left.\frac{d^n P}{dx^n}\right|_{x^*} > 0 \quad \text{where} \quad n \in \{4,6,8,\ldots\}$$

a minimum

$$\left.\frac{d^n P}{dx^n}\right|_{x^*} < 0 \quad \text{where} \quad n \in \{4,6,8,\ldots\}$$

a maximum

Univariate functions

Maximise a concave differentiable function f

NSC: x* is an optimal solution iff

$$\frac{\mathrm{df}}{\mathrm{dx}} = 0 \qquad \text{per } \mathrm{x} = \mathrm{x}^{*}$$

example

$$g(x) = 2x^3 - e^x \qquad g'(x) = 6x^2 - e^x = 0$$

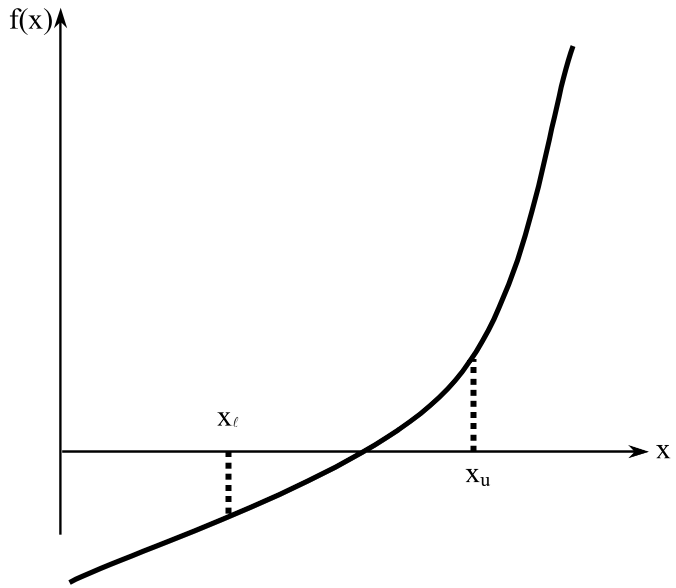
Univariate functions

If the equation to find x* can be solved analitically then the procedure is very straight forward, we solve it and we have found the optimum

But what happens if the function is too complex to be solved analitically ?

Basis of Bisection Method

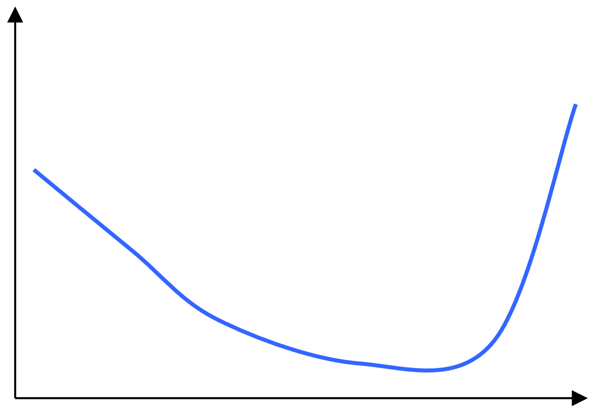
An equation $f(x)=0$, where $f(x)$ is a real continuous function, has at least one root between x_l and x_u if $f(x_l)*f(x_u) < 0$.



At least one root exists between the two points if the function is real, continuous, and changes sign.

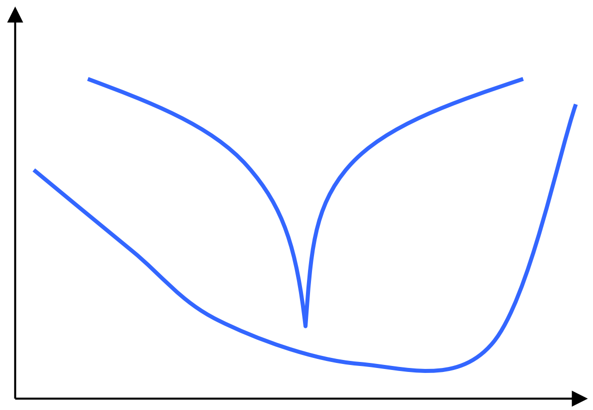
Unimodality

"An unimodal function consists of exactly one monotonically increasing and decreasing part"



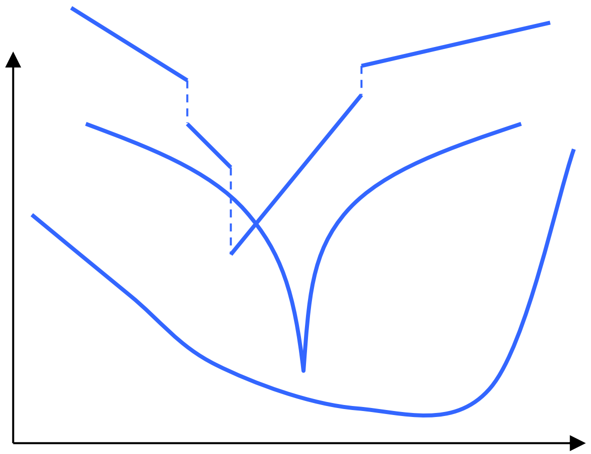
Unimodality

"An unimodal function consists of exactly one monotonically increasing and decreasing part"



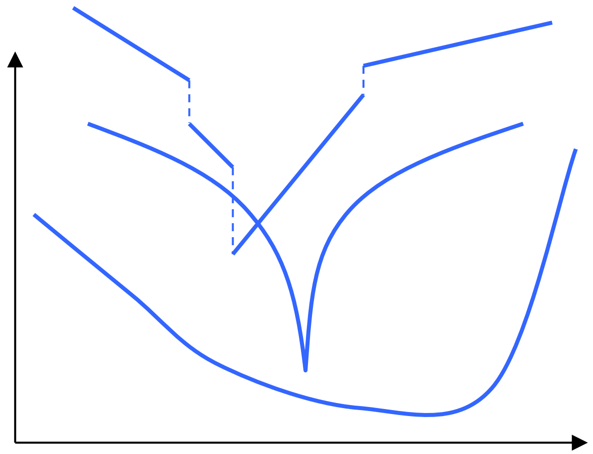
Unimodality

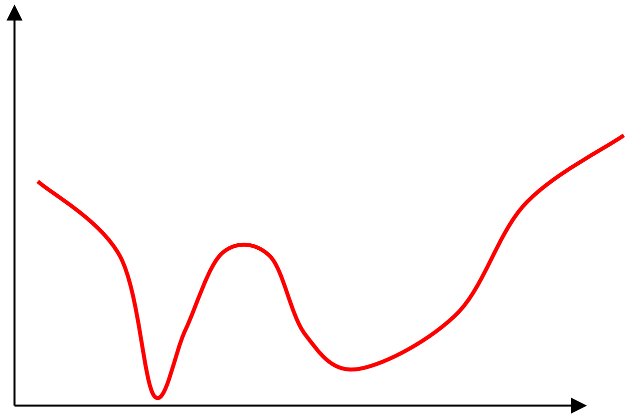
"An unimodal function consists of exactly one monotonically increasing and decreasing part"



Unimodality

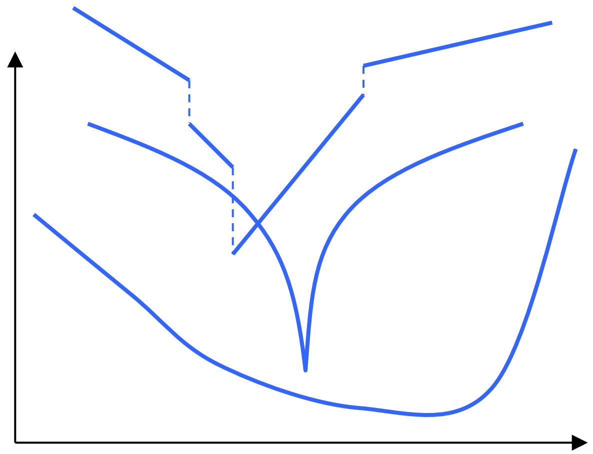
"An unimodal function consists of exactly one monotonically increasing and decreasing part"

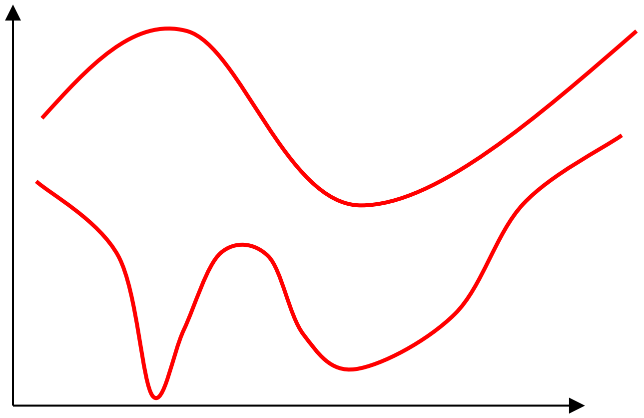




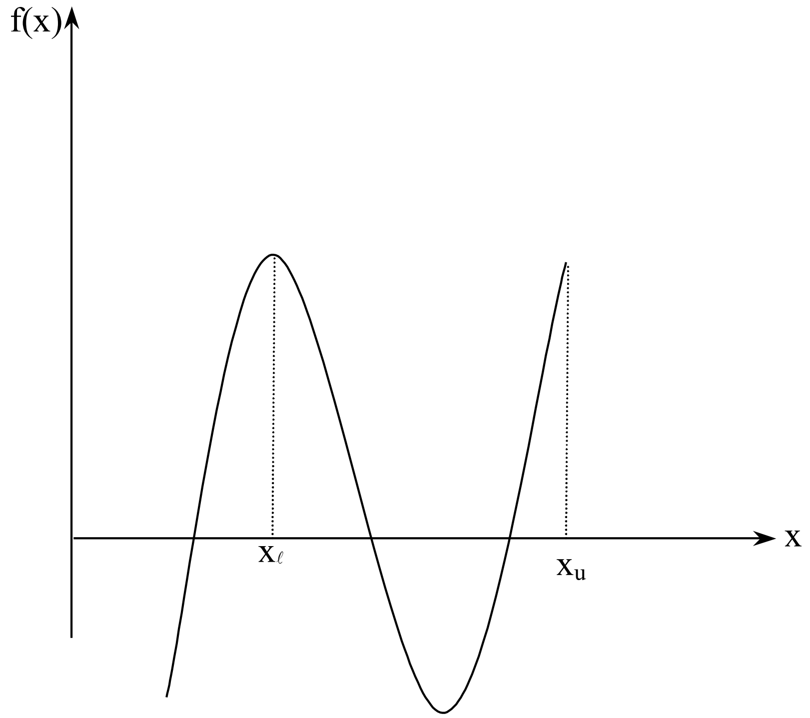
Unimodality

"An unimodal function consists of exactly one monotonically increasing and decreasing part"



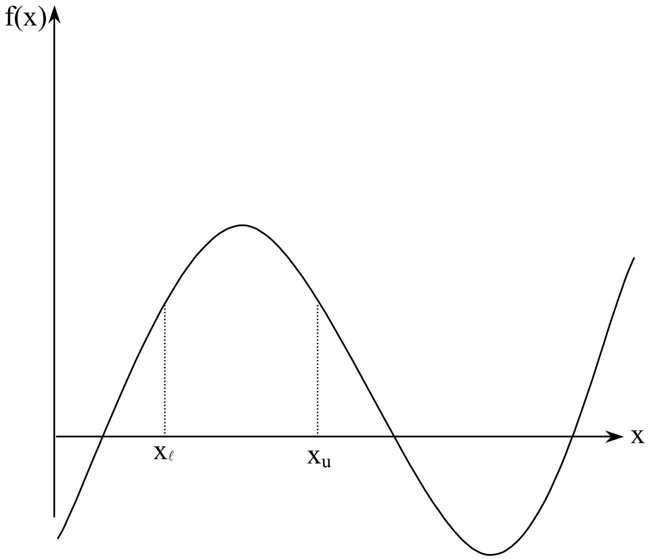


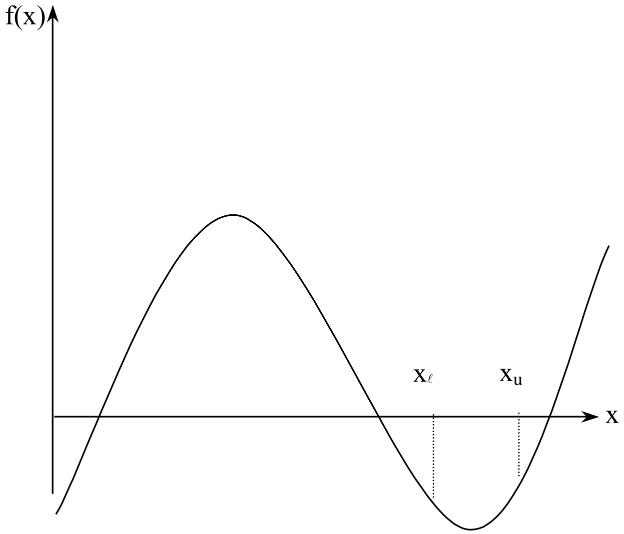
Basis of Bisection Method



If function $f(x)$ does not change sign between two points, roots of the equation $f(x)=0$ may still exist between the two points.

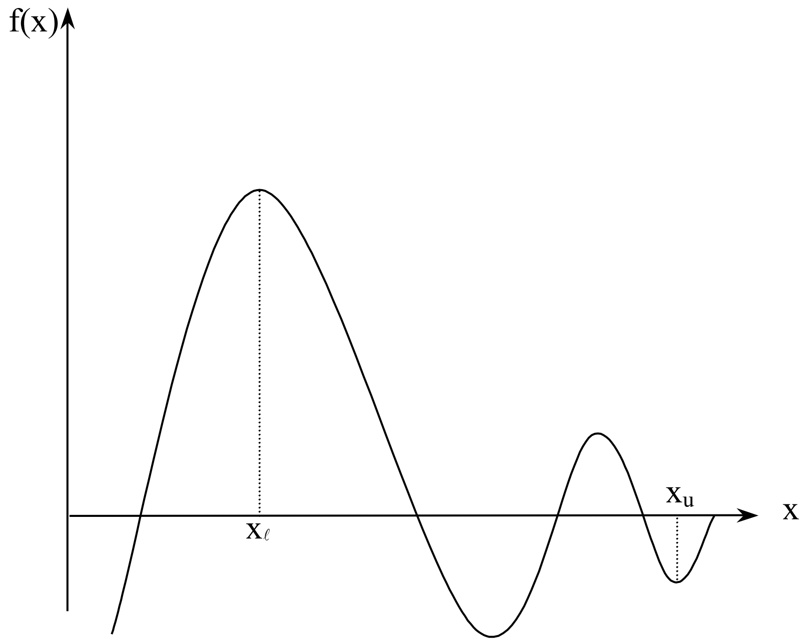
Basis of Bisection Method





If the function $f(x)$ does not change sign between two points, there may not be any roots for the equation $f(x)=0$ between the two points.

Basis of Bisection Method

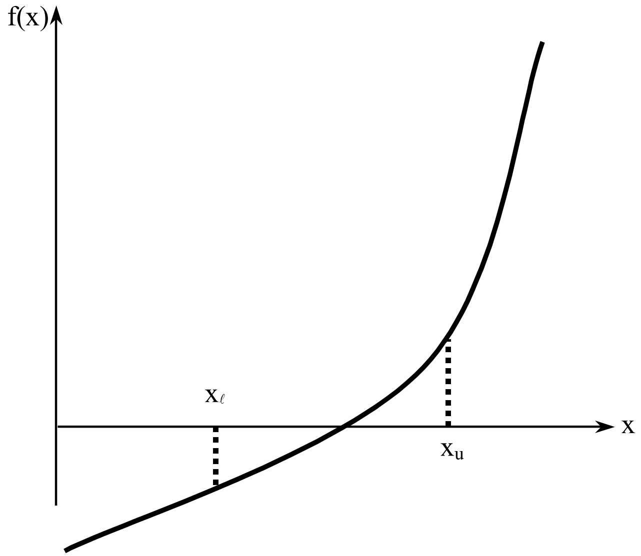


If the function $f(x)$ changes sign between two points, more than one root for the equation $f(x)=0$ may exist between the two points.

Algorithm for Bisection Method

Step 1

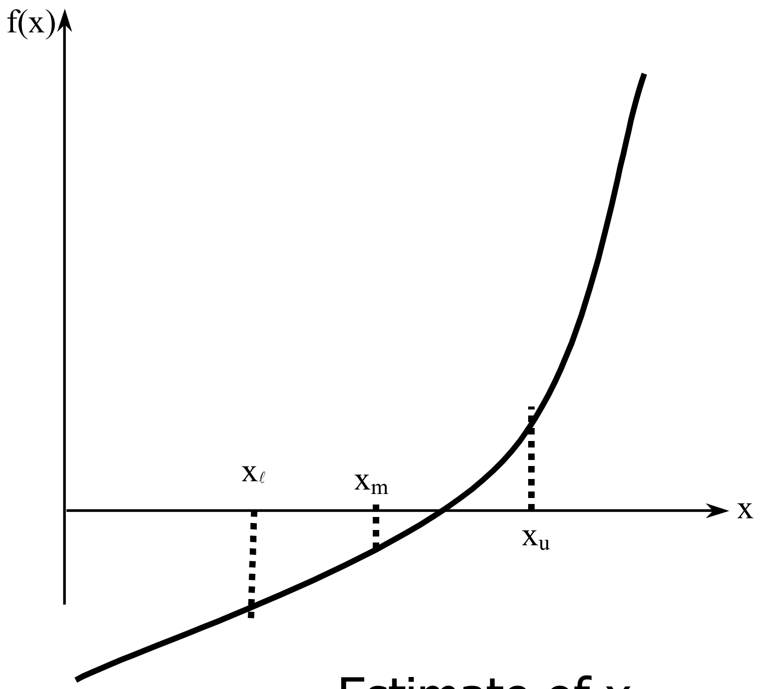
Choose x_ℓ and x_u as two guesses for the root such that $f(x_\ell)*f(x_u) < 0$, or in other words, $f(x)$ changes sign between x_ℓ and x_u.



Step 2

Estimate the root, x_m of the equation f (x) = 0 as the mid point between x_ℓ and x_u as

$$x_m = \frac{x_\ell + x_u}{2}$$



Estimate of x_m

Step 3

Now check the following

a) If $f(x_l)f(x_m)<0$, then the root lies between x_ℓ and x_m; then $x_\ell = x_\ell$; $x_u = x_m$.

b) If $f(x_l)f(x_m)>0$, then the root lies between x_m and x_u; then $x_\ell = x_m$; $x_u = x_u$.

c) If $f(x_l)f(x_m)=0$; then the root is x_m.

Stop the algorithm if this is true.

Step 4

Find the new estimate of the root

$$x_m = \frac{x_\ell + x_u}{2}$$

Find the absolute relative approximate error

$$\left|\in_a\right| = \left|\frac{x_m^{new} - x_m^{old}}{x_m^{new}}\right| \times 100$$

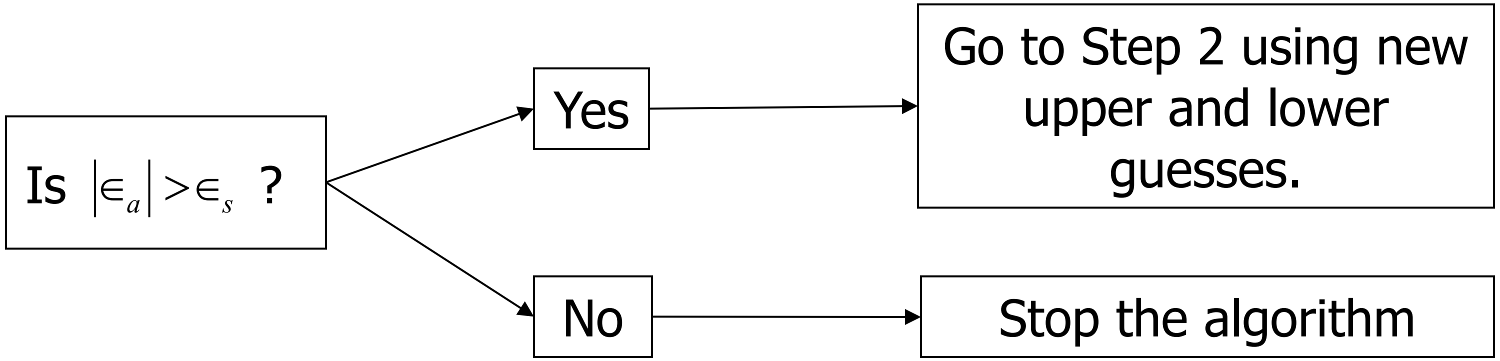
where

$$x_m^{old} = \text{previous estimate of root}$$

$$x_m^{new} = \text{current estimate of root}$$

Step 5

Compare the absolute relative approximate error $|\in_a|$ with the pre-specified error tolerance $\in_s$.

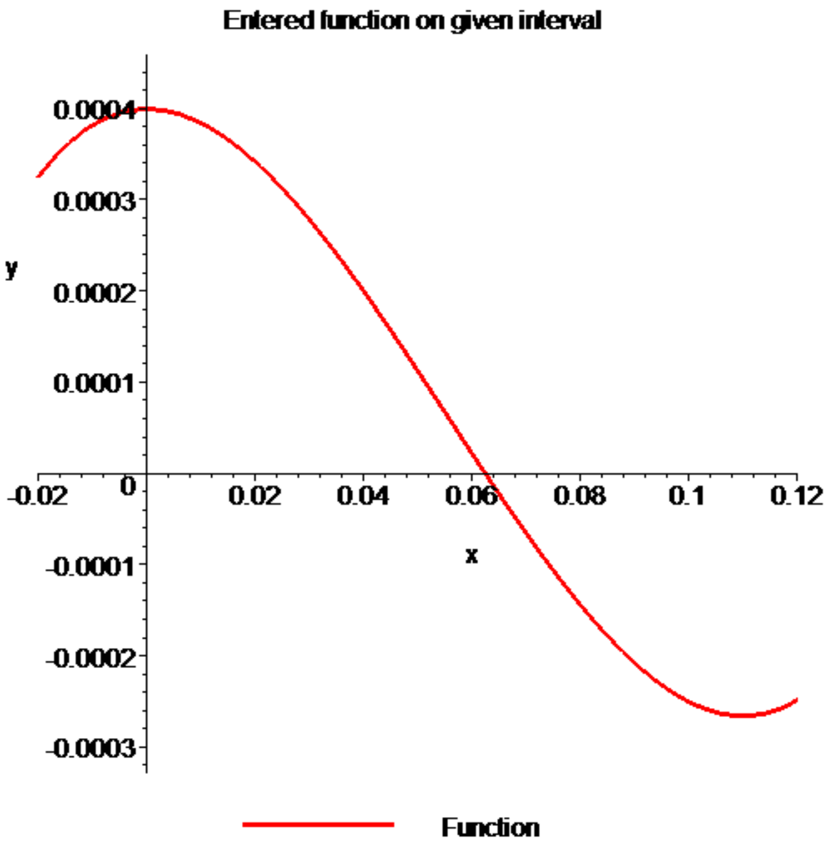


Note one should also check whether the number of iterations is more than the maximum number of iterations allowed. If so, one needs to terminate the algorithm and notify the user about it.

Example

$$f(x) = x^3 - 0.165x^2 + 3.993\times10^{-4}$$

To aid in the understanding of how this method works to find the root of an equation, the graph of f(x) is shown to the right



Graph of the function f(x)

Example 1 Cont.

Let us assume

$$x_\ell = 0.00$$

$$x_u = 0.11$$

Check if the function changes sign between x_ℓ and x_u .

$$f(x_l) = f(0) = (0)^3 - 0.165(0)^2 + 3.993\times10^{-4} = 3.993\times10^{-4}$$

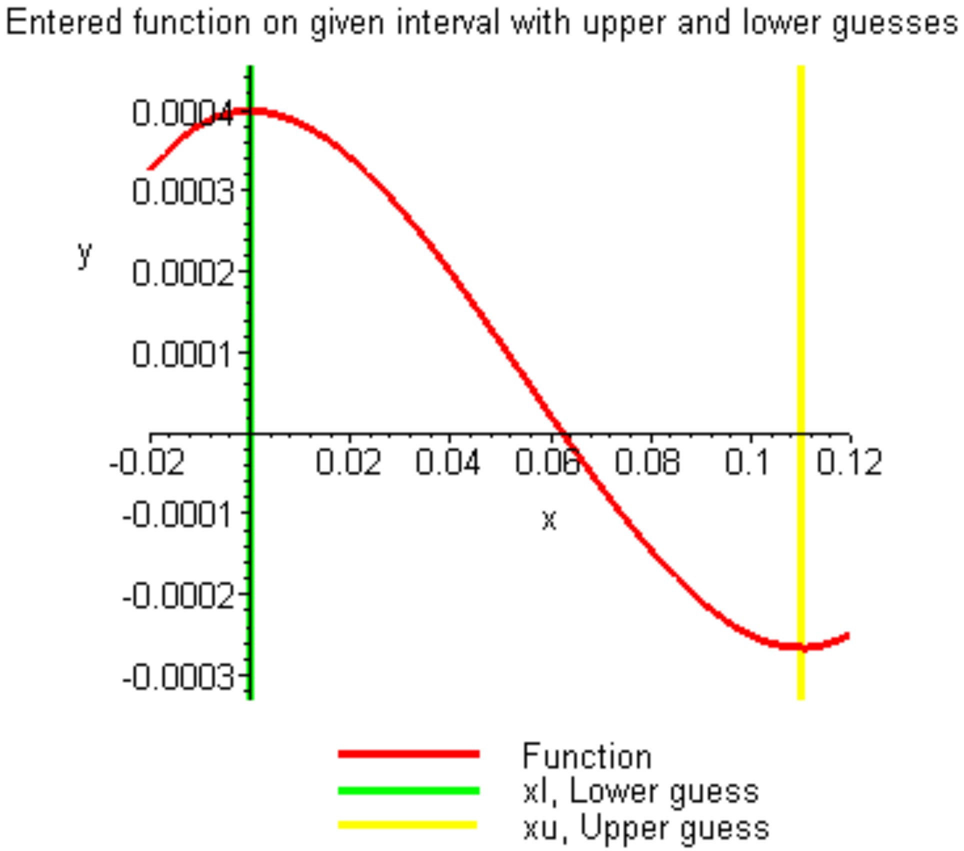
$$f(x_u) = f(0.11) = (0.11)^3 - 0.165(0.11)^2 + 3.993\times10^{-4} = -2.662\times10^{-4}$$

Hence

$$f(x_l)f(x_u) = f(0)f(0.11) = \left(3.993\times10^{-4}\right)\left(-2.662\times10^{-4}\right) < 0$$

So there is at least on root between x_ℓ and $x_{u,}$ that is between 0 and 0.11

Example 1 Cont.



Graph demonstrating sign change between initial limits

Example 1 Cont.

Iteration 1

The estimate of the root is

$$x_m = \frac{x_\ell + x_u}{2} = \frac{0+0.11}{2} = 0.055$$

$$f(x_m) = f(0.055) = (0.055)^3 - 0.165(0.055)^2 + 3.993\times10^{-4} = 6.655\times10^{-5}$$

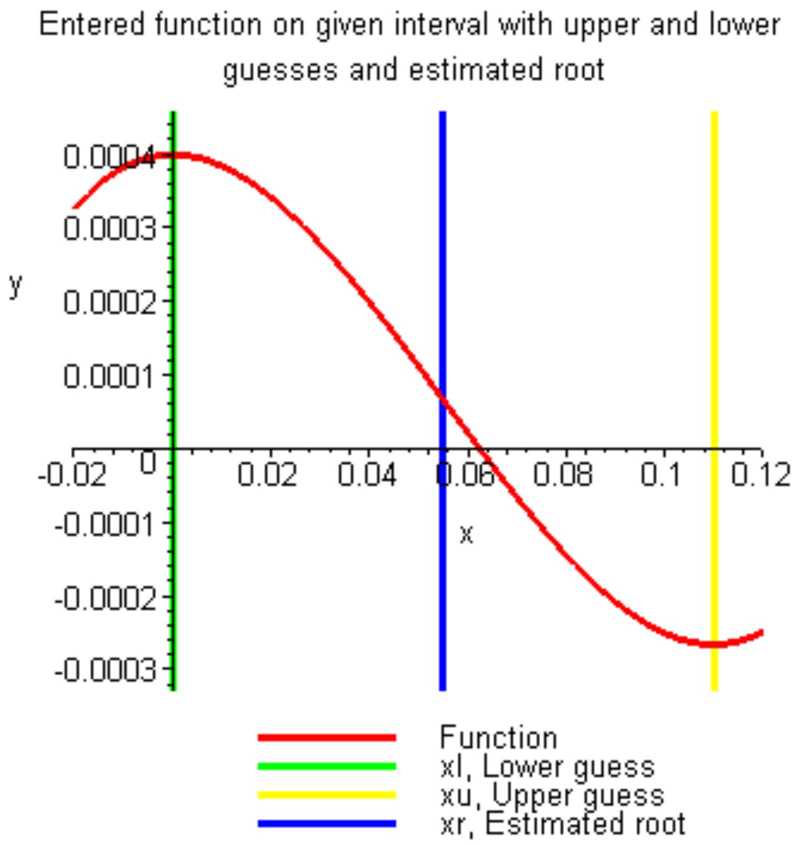
$$f(x_l)f(x_m) = f(0)f(0.055) = \left(3.993\times10^{-4}\right)\left(6.655\times10^{-5}\right) > 0$$

Hence the root is bracketed between x_m and x_u, that is, between 0.055 and 0.11. So, the lower and upper limits of the new bracket are

$$x_l = 0.055,\ x_u = 0.11$$

At this point, the absolute relative approximate error $\left|\in_a\right|$ cannot be calculated as we do not have a previous approximation.

Example 1 Cont.



Estimate of the root for Iteration 1

Example 1 Cont.

Iteration 2

The estimate of the root is $$x_m = \frac{x_\ell + x_u}{2} = \frac{0.055 + 0.11}{2} = 0.0825$$

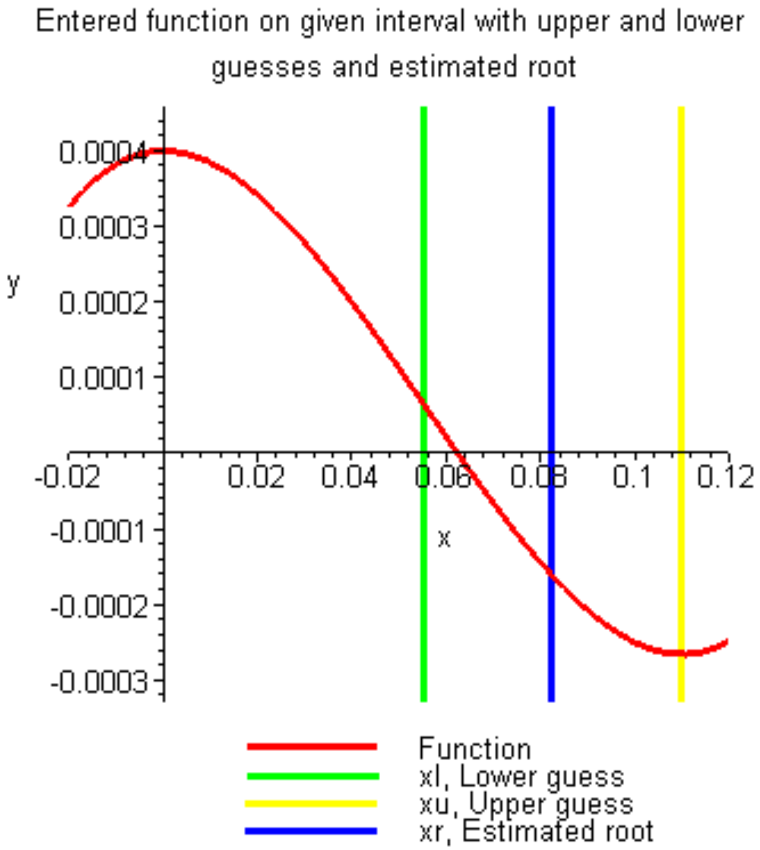
$$f(x_m) = f(0.0825) = (0.0825)^3 - 0.165(0.0825)^2 + 3.993\times 10^{-4} = -1.622\times 10^{-4}$$

$$f(x_l)f(x_m) = f(0.055)f(0.0825) = (-1.622\times 10^{-4})(6.655\times 10^{-5}) < 0$$

Hence the root is bracketed between x_ℓ and x_m, that is, between 0.055 and 0.0825. So, the lower and upper limits of the new bracket are

$$x_l = 0.055,\ x_u = 0.0825$$

Example 1 Cont.



Estimate of the root for Iteration 2

Example 1 Cont.

The absolute relative approximate error $\left|\in_a\right|$ at the end of Iteration 2 is

$$\left|\in_a\right| = \left|\frac{x_m^{new} - x_m^{old}}{x_m^{new}}\right| \times 100$$

$$= \left|\frac{0.0825 - 0.055}{0.0825}\right| \times 100$$

$$= 33.333\%$$

Suppose now that we want to find the root with a tolerance error lower than 5%, then we have to continue with the next iteration

Example 1 Cont.

Iteration 3

The estimate of the root is $x_m = \dfrac{x_\ell + x_u}{2} = \dfrac{0.055 + 0.0825}{2} = 0.06875$

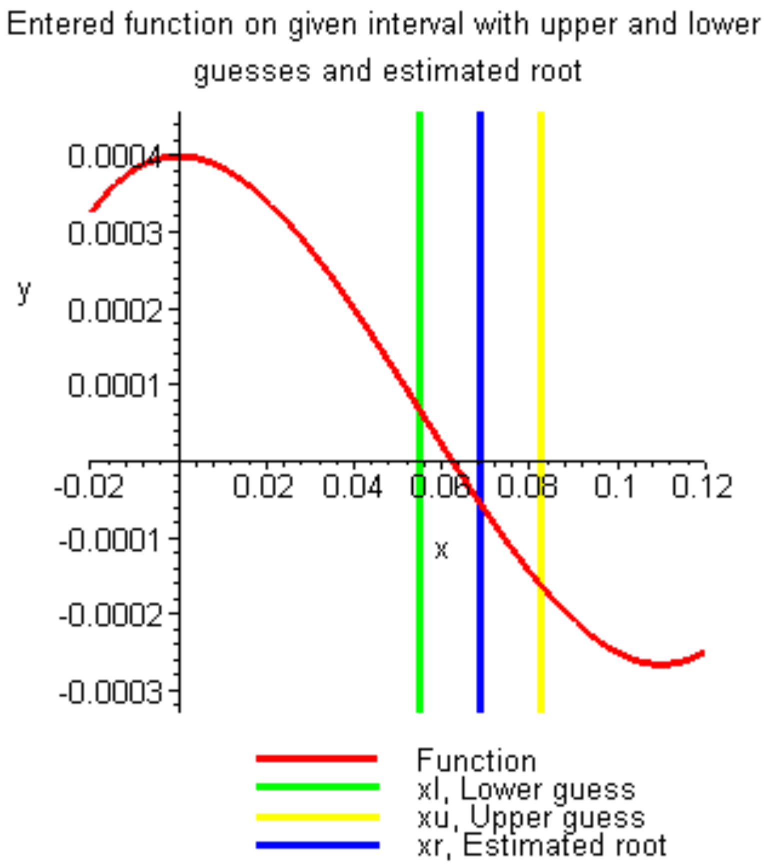
$$f(x_m) = f(0.06875) = (0.06875)^3 - 0.165(0.06875)^2 + 3.993\times10^{-4} = -5.563\times10^{-5}$$

$$f(x_l)f(x_m) = f(0.055)f(0.06875) = (6.655\times10^{-5})(-5.563\times10^{-5}) < 0$$

Hence the root is bracketed between x_ℓ and x_m, that is, between 0.055 and 0.06875. So, the lower and upper limits of the new bracket are

$$x_l = 0.055,\ x_u = 0.06875$$

Example 1 Cont.



Estimate of the root for Iteration 3

Example 1 Cont.

The absolute relative approximate error $|\in_a|$ at the end of Iteration 3 is

$$|\in_a| = \left| \frac{x_m^{new} - x_m^{old}}{x_m^{new}} \right| \times 100$$

$$= \left| \frac{0.06875 - 0.0825}{0.06875} \right| \times 100$$

$$= 20\%$$

Still not enough as the absolute relative approximate error is greater than 5%.

Example Cont.

Root of f(x)=0 as function of number of iterations for bisection method.

Iteration	x_ℓ	x_u	x_m	$\left\|\in_a\right\|$ %	$f(x_m)$
1	0.00000	0.11	0.055	----------	6.655×10^{-5}
2	0.055	0.11	0.0825	33.33	-1.622×10^{-4}
3	0.055	0.0825	0.06875	20.00	-5.563×10^{-5}
4	0.055	0.06875	0.06188	11.11	4.484×10^{-6}
5	0.06188	0.06875	0.06531	5.263	-2.593×10^{-5}
6	0.06188	0.06531	0.06359	2.702	-1.0804×10^{-5}
7	0.06188	0.06359	0.06273	1.370	-3.176×10^{-6}
8	0.06188	0.06273	0.0623	0.6897	6.497×10^{-7}
9	0.0623	0.06273	0.06252	0.3436	-1.265×10^{-6}
10	0.0623	0.06252	0.06241	0.1721	-3.0768×10^{-7}

Advantages

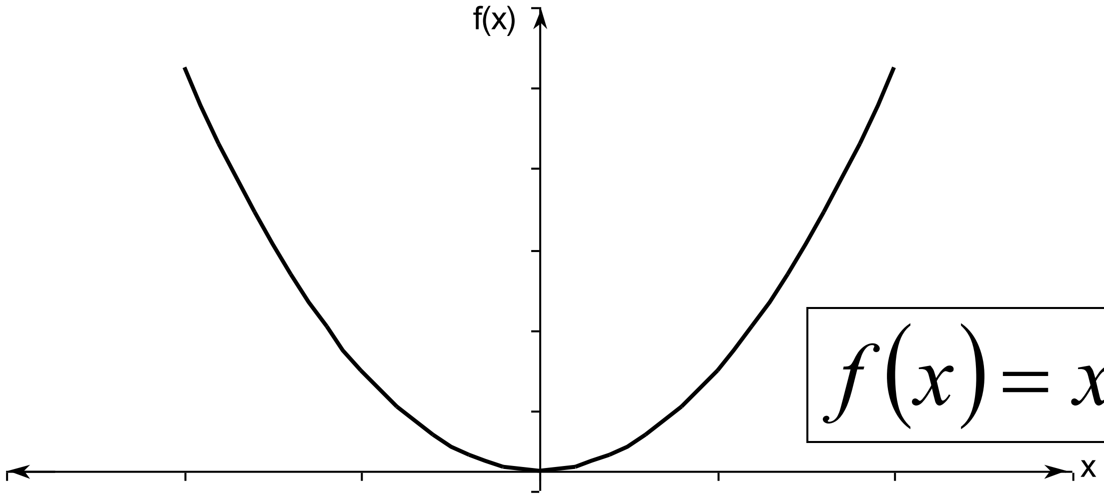
- Always convergent
- The root bracket gets halved with each iteration - guaranteed.

Drawbacks

- Slow convergence
- If one of the initial guesses is close to the root, the convergence is slower

Drawbacks (continued)

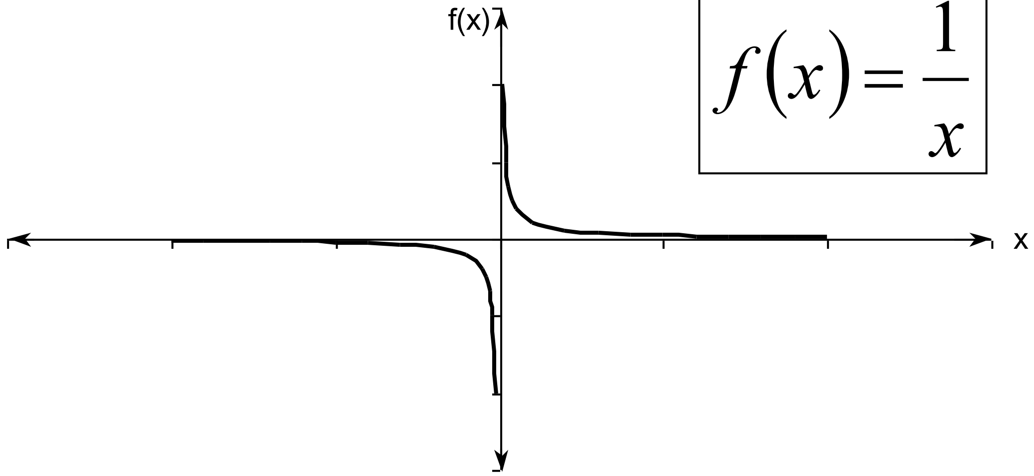
- If a function f(x) is such that it just touches the x-axis it will be unable to find the lower and upper guesses.



$$f(x) = x^2$$

Drawbacks (continued)

- Function changes sign but root does not exist



$$f(x) = \frac{1}{x}$$